Coupled modified baker's transformations for the Ising model

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An invertible coupled map lattice is proposed for the Ising model. Each elemental map is a modified baker's transformation, which is a two-dimensional map of X and Y. The time evolution of the spin variable is memorized in the binary representation of the Y variable. The temporal entropy and time correlation of the spin variable are calculated from the snapshot configuration of the Y variables. [S1063-651X(99)07012-9]

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Coupled map lattice models have been used to study chaotic spatiotemporal patterns [1–3]. We proposed a coupled map lattice which exhibits a thermodynamic phase transition equivalent to the Ising model [4–6]. In the previous model, each elemental map is the Bernoulli shift. The Bernoulli shift is a one-dimensional map and it is not invertible. We propose an invertible coupled map lattice for the Ising model in this paper. The invertibility implies that the state in the past can be known from the information of the present state. In our model, the spin state in the past is memorized especially in a simple manner. It is an example of simple chaotic dynamical systems which has nontrivial thermodynamic properties.

Each elemental map is a modified baker's transformation:

$$X_{n+1}=\frac{2}{1+a}(X_n+1)-1,$$

$$Y_{n+1}=\frac{1}{2}(Y_n+1) \quad \text{for} \quad -1<X_n<a,$$

$$X_{n+1}=\frac{2}{1-a}(X_n-1)+1,$$

$$Y_{n+1}=\frac{1}{2}(Y_n-1) \quad \text{for} \quad a<X_n<1, \tag{1}$$

where a is a parameter satisfying $-1<a<1$. It is a two-dimensional map from a square region $[-1,1]\times[-1,1]$ to the same square region. The two-dimensional map is reduced to the original baker's transformation for $a=0$. The mapping of X is equivalent to the Bernoulli shift. The Bernoulli shift has a uniform invariant measure $\rho(X)=1/2$ over $-1<X<1$. A spin variable S_n is defined as $S_n=\mathrm{sgn}(X_{n+1}-X_n)$, that is, $S_n=1$ for $X_n<a$ and $S_n=-1$ for $X_n>a$. The mean value of S_n is a and its time correlation is $\langle S_n S_m\rangle=\delta_{n,m}$. The spin variable S_n is also written as $S_n=\mathrm{sgn}(Y_{n+1})$.

We can construct various coupled map systems using the elemental map. A globally coupled map model is constructed as

$$X_{n+1}^i=\frac{2}{1+a_n}(X_n^i+1)-1,$$

$$Y_{n+1}^i=\frac{1}{2}(Y_n^i+1) \quad \text{for} \quad -1<X_n^i<a_n,$$

$$X_{n+1}^i=\frac{2}{1-a_n}(X_n^i-1)+1,$$

$$Y_{n+1}^i=\frac{1}{2}(Y_n^i-1) \quad \text{for} \quad a_n<X_n^i<1, \tag{2}$$

where $i=1,2,\ldots,N$. The parameter a_n is expressed as

$$a_n=\tanh\left(K\sum_{i=1}^{N} S_{n-1}^i/N\right), \tag{3}$$

where K is a coupling constant and $S_n^i=\mathrm{sgn}(X_{n+1}^i-X_n^i)$. The uniform invariant measure $\rho=1/2$ is maintained even if the parameter a_n is temporally fluctuating. The mean value of S_n^i is approximated at a_n. The order parameter defined as $M_n=1/N\sum_{i=1}^N S_n^i$ satisfies

$$M_n=1/N\sum_{i=1}^{N} S_n^i=a_n=\tanh(KM_{n-1}), \tag{4}$$

when $N\to\infty$. A phase transition occurs at $K=1$.

A two-dimensional coupled map lattice is constructed as

$$X_{n+1}^{i,j}=\frac{2}{1+a_n^{i,j}}(X_n^{i,j}+1)-1,$$

$$Y_{n+1}^{i,j}=\frac{1}{2}(Y_n^{i,j}+1) \quad \text{for} \quad -1<X_n^{i,j}<a_n^{i,j},$$

$$X_{n+1}^{i,j}=\frac{2}{1-a_n^{i,j}}(X_n^{i,j}-1)+1,$$

$$Y_{n+1}^{i,j}=\frac{1}{2}(Y_n^{i,j}-1) \quad \text{for} \quad a_n^{i,j}<X_n^{i,j}<1, \tag{5}$$

where $1\leqslant i\leqslant L$ and $1\leqslant j\leqslant L$ denote a lattice point in the $L\times L$ square lattice. The parameter $a_n^{i,j}$ is variable expressed as

$$a_n^{i,j}=\tanh\left\{\frac{K}{4}(S_{n-1}^{i+1,j}+S_{n-1}^{i-1,j}+S_{n-1}^{i,j+1}+S_{n-1}^{i,j-1})\right\}, \quad (6)$$

where K is a coupling constant. The updating is performed alternatively for even lattice points (where $i+j$ is even) and odd lattice points. In this case, the probability distribution $p_n(\{m^{i,j}\})$ that the spin configuration $\{S_n^{i,j}\}$ takes $\{m^{i,j}\}$ at time n obeys a master equation

$$p_n(\{m^{i,j}\})=\sum_{m'^{i,j}} p_{n-1}(\{m'^{i,j}\})w(\{m'^{i,j}\}\to\{m^{i,j}\}), \quad (7)$$

where $w(\{m'^{i,j}\}\to\{m^{i,j}\})$ is the transition probability. The equilibrium distribution p_{eq} is obtained as

$$p_{eq}(\{m^{i,j}\})\propto\exp\left\{K/8\sum_{i,j} m^{i,j}(m^{i+1,j}+m^{i-1,j}+m^{i,j+1}+m^{i,j-1})\right\}, \quad (8)$$

which is equivalent to the equilibrium distribution of the two-dimensional Ising model. The Ising system exhibits a phase transition at $K=2\ln(1+\sqrt{2})$ for $N=L^2\to\infty$. This phase transition was numerically checked in Ref. [4] for the coupled Bernoulli map lattice. We consider mainly the role of Y variables hereafter in the coupled modified baker's transformations.

The coupled map lattice is invertible. The time-reversal mapping for the two-dimensional model is written as

$$Y_n^{i,j}=2Y_{n+1}^{i,j}-1,$$

$$X_n^{i,j}=\frac{1+a_n^{i,j}}{2}(X_{n+1}^{i,j}+1)-1 \quad \text{for} \quad 0<Y_{n+1}^{i,j}<1,$$

$$Y_n^{i,j}=2Y_{n+1}^{i,j}+1,$$

$$X_n^{i,j}=\frac{1-a_n^{i,j}}{2}(X_{n+1}^{i,j}-1)+1 \quad \text{for} \quad -1<Y_{n+1}^{i,j}<0. \quad (9)$$

The parameter $a_n^{i,j}$ is given by Eq. (6) where $S_{n-1}^{i,j}=\text{sgn}(Y_n^{i,j})$. The invertibility implies that the state in the past can be known from the information of the present state. The memory of the past state is stored in the Y variables. A variable $y=(Y+1)/2$ is a real number between 0 and 1, therefore y can be represented by a binary notation as $0.s_1s_2s_3s_4s_5\cdots$, where s_k is 0 or 1. The forward time evolution of Y by Eq. (5) is represented as a shift of the sequence $\{s_k\}$ in the right direction: $0.s_1s_2s_3s_4s_5\cdots\to0.s_0s_1s_2s_3s_4s_5\cdots$. The newly added bit s_0 is equal to $(S_n+1)/2$. In other words, the binary notation $0.s_0s_1s_2s_3s_4\cdots$ at step $n+1$ implies that $2s_k-1$ is the spin variable S_{n-k} at step $n-k$. The history of the spin variable $\{S_{n-k}^{i,j}\}$ is represented in the binary notation of the $y^{i,j}$ variables at a certain step. The inverted time evolution of Y by Eq. (9) is represented as a shift of $\{s_k\}$ in the inverse direction: $0.s_1s_2s_3s_4s_5\cdots\to0.s_2s_3s_4s_5\cdots$. We will study several statistical properties of the time sequence of the spin variables using the snapshot profiles of Y variables.

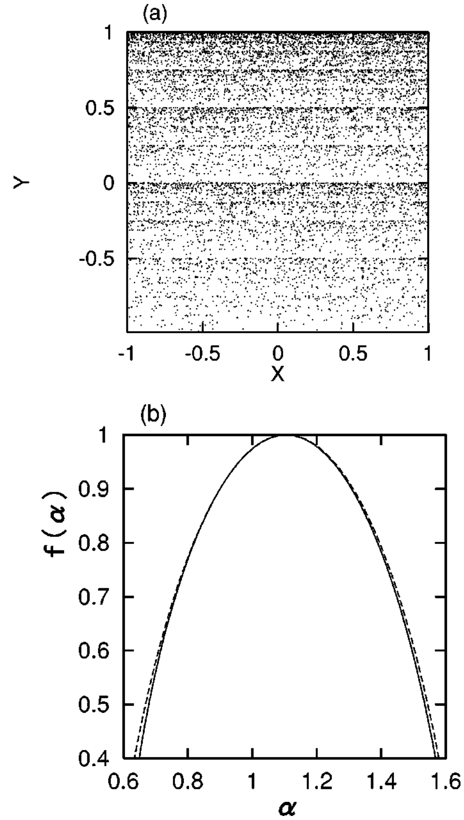

FIG. 1. (a) Snapshot pattern of (X^i,Y^i) at $K=1.05$ for the globally coupled map model (2). (b) The solid line is the $f(\alpha)$ spectrum for the globally coupled map model and the dashed line is that for the modified baker's transformation with $a=0.3707$.

We first study a fine structure of the snap-shot profile (X_n^i,Y_n^i) of the globally coupled system (2). For the globally coupled system with $N=\infty$, the order parameter decays to zero and the spin configuration is random for $K<1$. For $K>1$, the order parameter takes a constant nonzero value M. The elemental map is a modified baker's transformation with a constant parameter $a=\tanh KM$. The entropy per one step for the time sequence of S_k is $-p\ln p-(1-p)\ln(1-p)$ where $p=(1+a)/2$. Figure 1(a) displays a snapshot of (X^i,Y^i) for $i=1,2,\ldots,N$ at $K=1.05$ and $N=10\,000$. A random initial condition for X_0^i and Y_0^i is assumed and S_0^i is all 1. We have calculated the generalized dimension D_q and its Legendre transform $f(\alpha)$ spectrum to characterize the multifractal pattern [7]. D_q is defined as

$$D_q=\lim_{l\to0}\frac{1}{q-1}\frac{\ln\sum_{i=1}^{N_0}p_i^q}{\ln l}, \quad (10)$$

where the interval $[0,1]$ is divided into N_0 intervals, $l=1/N_0$ is the size of the divided small intervals and p_i is the probability that y^i locates in the ith interval. The solid line in Fig. 1(b) displays the $f(\alpha)$ spectrum which have been numerically obtained from the snap-shot profiles of the globally coupled map with $N=10\,000$ and l is assumed to be 1/1024. The dashed line in Fig. 1(b) is the $f(\alpha)$ spectrum for the strange attractor of the elemental modified baker's transformation with a constant a:

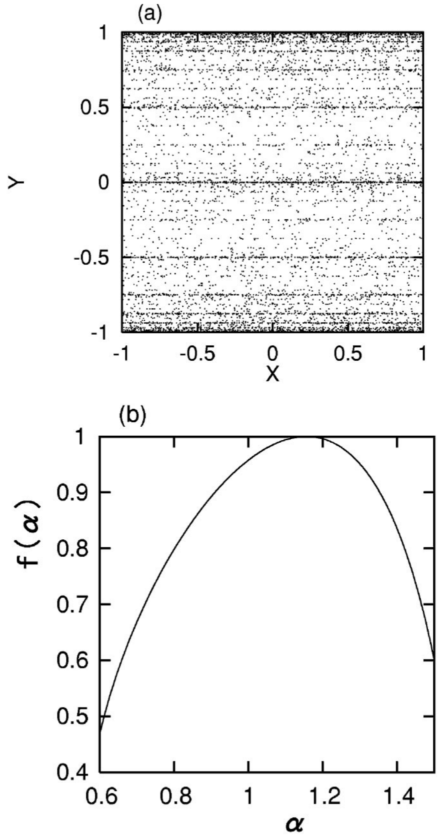

FIG. 2. (a) Snapshot pattern of $(X^{i,j},Y^{i,j})$ at $K=1.6$ for the two-dimensional coupled map lattice (5). (b) The solid line is the $f(\alpha)$ spectrum for the two-dimensional coupled map lattice at $K=1.6$.

$$f(r)=-\frac{r\ln r+(1-r)\ln(1-r)}{\ln 2},$$

$$\alpha=-\frac{r\ln p+(1-r)\ln(1-p)}{\ln 2}, \tag{11}$$

where r is a parameter between 0 and 1 and $p=(1+a)/2$. The constant parameter a is given by $a=M=\tanh(KM)=0.3707$ for $K=1.05$. The peak value of $f(\alpha)$ is equal to the capacity dimension D_0, and $D_0=1$ for the modified baker's transformation. But the information dimension $D_1=\{-p\ln p-(1-p)\ln(1-p)\}/\ln 2$ is smaller than 1. A good agreement is seen irrespective of the small fluctuation for the order parameter M_n.

If the interval $[0,1]$ is divided into 2^n intervals ($l=1/2^n$), each small interval corresponds respectively to a binary notation $0.s_1s_2\cdots s_n$. For example, the interval $[3/8,1/2]$ is denoted as 0.011. The probability p_i for each interval is therefore equal to the probability $P(S_1,S_2,\ldots,S_n)$ of the corresponding time sequence of the spin: $S_1,S_2,\ldots,S_n$. The quantity $\ln[\Sigma_{\{S_k\}}\{p(S_1,S_2,\ldots,S_n)\}^q]/(1-q)$ is called the Renyi entropy for the spin sequence. The Renyi entropy at $q=1$ is the Shannon entropy and it is proportional to the length n of the sequence. The Renyi entropy over the step number n is represented as $D_q\ln 2$ in our model from Eq. (10). The Shannon entropy over n is equal to the KS entropy H_{KS} of the Bernoulli map which is the mapping for the X variable. The mean area-expansion rate in the X direction is equal to the KS entropy. The area contraction rate in the Y direction is $\ln 2$. The total area-contraction rate is $\ln 2-H_{KS}=\ln 2(1-D_1)$. It is the entropy production rate or the information loss rate for our dynamical system.

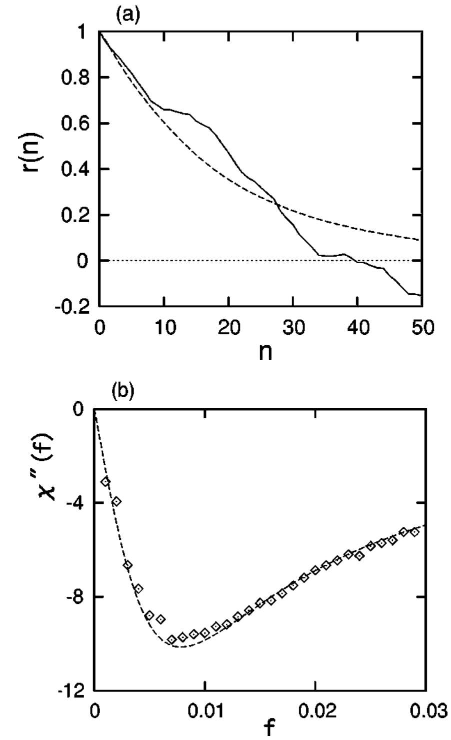

FIG. 3. (a) Correlation function $r(n)=\langle M_{m-n}M_m\rangle/\langle M_m^2\rangle$ at $K=1.45$ for the two-dimensional coupled map lattice. (b) The marks denotes $\chi''(f)$ calculated by Eq. (15) and the dashed line is $\chi''(f)$ given by Eq. (14).

We can calculate similar quantities for the two-dimensional coupled map lattice (5). Figure 2(a) displays a snapshot pattern of $(X^{i,j},Y^{i,j})$ for $L=100$ and $K=1.6$. The pattern seems to be multifractal and it is nearly symmetric with respect to $Y=0$, since the parameter $K<K_c$. Figure 2(b) displays the $f(\alpha)$ spectrum. The peak of $f(\alpha)$ is 1 as in Fig. 1(b). The KS entropy is a spatiotemporal entropy per one step and one site and is defined in our coupled map lattice as

$$H_{KS}=-\lim_{n\to\infty,N\to\infty}\{1/(nN)\}\times\sum_{\{S_k^{i,j}\}}p(\{S_1^{i,j}\},\{S_2^{i,j}\},\ldots,\{S_n^{i,j}\})\times\ln p(\{S_1^{i,j}\},\{S_2^{i,j}\},\ldots,\{S_n^{i,j}\}).$$

On the other hand the temporal entropy per one step and one site is given by

$$H_S=-\lim_{n\to\infty,N\to\infty}\{1/(nN)\}\times\sum_{S_k^{i,j},i,j}p(S_1^{i,j},S_2^{i,j},\ldots,S_n^{i,j})\ln p(S_1^{i,j},S_2^{i,j},\ldots,S_n^{i,j}).$$

That is, H_{KS} is the information rate per one site for the time sequence of the spin configuration $\{S^{i,j}\}$ and H_S is the average value of the information rate for the spin sequence at a certain site. The spatial mutual information is taken into consideration in the KS entropy, however, it is not taken into consideration in H_S, therefore, H_S is generally larger than H_{KS}. The temporal entropy per one step H_S is estimated as $D_1 \ln 2 \sim 0.575$, since the information dimension is $D_1 \sim 0.83 < 1$. It is surely larger than the numerically obtained KS entropy $H_{KS} = 0.358$.

The time sequence of the order parameter can be also calculated from the snapshot pattern of $Y^{i,j}$. For example, the magnetization $M_n = \Sigma S_n^{i,j}/N$ at step n is calculated as $M_n = 2N_0/N - 1$ where N_0 is the number of sites at which $Y_{n+1}^{i,j} > 0$, and the magnetization M_{n-1} is equal to $2N_1/N - 1$ where N_1 is the number of sites at which $-1/2 < Y_{n+1}^{i,j} < 0$ or $1/2 < Y_{n+1}^{i,j} < 1$. The time correlation of magnetization can be calculated as $\langle M_{m-n} M_m \rangle = \Sigma_{l=1}^{L_0} M_{m-n-l+1} M_{m-l+1}/L_0$. Figure 3(a) displays the time correlation function $r(n) = \langle M_{m-n} M_m \rangle / \langle M_m^2 \rangle$ for $K = 1.45$ and $L_0 = 100$. The solid line is calculated from a snapshot profile and the dashed line denotes the long-time average. The averaged correlation function decays exponentially as λ^n where $\lambda \sim 0.952$.

The time correlation of the order parameter is related to the response function of the order parameter for the time-periodic perturbation. Such perturbation can be given by changing the parameter $a_n^{i,j}$ as

$$a_n^{i,j} = \tanh\left\{ \frac{K}{4} (S_{n-1}^{i+1,j} + S_{n-1}^{i-1,j} + S_{n-1}^{i,j+1} + S_{n-1}^{i,j-1}) + b \cos(2\pi f n) \right\}, \tag{12}$$

where b is the amplitude and f is the frequency of the perturbation. A simple linear map with a similar time periodic perturbation is assumed to be

$$M_n = \lambda M_{n-1} + b \cos(2\pi f n). \tag{13}$$

The order parameter M decays exponentially as λ^n when $b = 0$. The linear mapping can be solved as $M_n \sim \mathrm{Re}[b \exp(i 2\pi f n)/\{1 - \lambda \exp(-i 2\pi f)\}]$ for $0 < \lambda < 1$. The complex susceptibility for the time periodic perturbation is therefore expressed as

$$\chi'(f) + i\chi''(f) = \frac{1 - \lambda \cos(2\pi f) - i\lambda \sin(2\pi f)}{1 + \lambda^2 - 2\lambda \cos(2\pi f)}. \tag{14}$$

On the other hand, the complex susceptibility can be numerically estimated from the time sequence of the spin configuration as

$$\chi'(f) + i\chi''(f) = \frac{2}{N n_1 b} \sum_{n=1}^{n_1} \sum_{i,j} S_n^{i,j} \exp(-i 2\pi f n), \tag{15}$$

where n_1 is a step number to calculate a time average. The numerical estimate of $\chi''(f)$ for the coupled modified baker's transformations with parameter $a_n^{i,j}$ by Eq. (12) is shown in Fig. 3(b) by the marks for various f at $K = 1.45$, $n_1 = 40\,000$ and $b = 0.005$. The dashed line denotes $\chi''(f)$ in Eq. (14) with $\lambda = 0.952$, which is a good approximation.

In summary we have proposed an invertible coupled map lattice for the Ising model. The time sequence of the spin configuration is memorized in the binary representation of the Y variables. The temporal entropy of spin variables and the time correlation of the order parameter have been calculated from the snapshot data of the Y variables. This coupled map lattice is an instructive model that connects the chaotic dynamics and the statistical mechanics.

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